Topological order from quantum loops and nets

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Abstract

I define models of quantum loops and nets that have ground states with topological order. These make possible excited states comprised of deconfined anyons with non-abelian braiding. With the appropriate inner product, these quantum loop models are equivalent to net models whose topological weight involves the chromatic polynomial. A simple Hamiltonian preserving the topological order is found by exploiting quantum self-duality. For the square lattice, this Hamiltonian has only four-spin interactions.

1 Introduction

Condensed-matter systems with anyonic excitations have been the subject of intense study recently, one reason being their potential application to topological quantum computation [1]. Anyonic excitations can occur in many-body systems with *fractionalization*, i.e. where the quasiparticles of the system have quantum numbers which are fractions of the underlying "fundamental" excitations. For example, in the fractional quantum Hall effect, quasiparticles of charge $e/3$ and $e/5$ have been observed, even though the system is comprised purely of electrons. The Hall effect has given the only unambiguous experimental realization of fractionalized excitations, but since this behavior is so novel, and the consequences potentially so dramatic, it is important to find different sorts of models exhibiting this behavior. Thus considerable theoretical attention also has been devoted to finding magnetic models exhibiting anyonic excitations.

It is now well understood how abelian fractionalized excitations can occur in relatively simple spin systems, e.g. the "toric code" [2], and quantum dimer models [3]. For topological quantum computation, however, it is essential that the excitations not only be fractionalized, but also have non-abelian braiding. Magnetic models with this behavior are much trickier to find. For a model containing non-abelian anyons to be mathematically consistent, the braiding and fusing properties of the anyons must satisfy very intricate constraints [1, 4]. At a glance, one might think it impossible for the quasiparticles in any physical system to satisfy these properties. Nevertheless, we know from the fractional quantum Hall effect that these properties do indeed arise as a consequence of well-motivated (and well-checked numerically) model wavefunctions [5]. It is thus reasonable to hope that non-abelian excitations can arise in a physically-relevant system.

At the moment, however, magnetic models with non-abelian anyons do not arise as naturally as those coming from the Hall effect or other chiral systems. Thus attempts to find them have mainly been reverse engineering [6]. Namely, first one finds a mathematically-consistent set of

braiding and fusion relations; in mathematics this is called a modular tensor category, while in physics these rules are familiar from two-dimensional conformal field theory. Then one attempts to fine-tune a Hamiltonian to obtain this braiding and fusing. With sufficiently complicated interactions, it is indeed possible to build this structure into a Hamiltonian, for example in "string-net" models [7]. Moreover, in at least one special case, it is also possible to find a simple (although highly fine-tuned) Hamiltonian whose excitations involve Majorana fermions, and when time-reversal symmetry is broken, non-abelian braiding [8]. These Hamiltonians therefore provide valuable proofs of principle.

Fractionalized excitations (abelian and non-abelian) in magnetic models generally arise because they have non-trivial topological characteristics. Braiding is an inherently topological property – the statistics depends only on the topology of the paths in configuration space for taking the particles around each other. In order to find more realistic models, it is therefore desirable to find ones which incorporate directly the *topological* properties of non-abelian braiding, not only the algebraic ones. To this end, substantial attention has been given to *quantum loop models* [6]. These have the nice feature that the necessary topological properties of non-abelian braiding are built in from the beginning.

More precisely, an essential property for having deconfined anyons is that the ground state can be expressed as a superposition of states, each of which is characterized by a set of geometric objects. In quantum loop models each configuration in the ground-state sum is described by a set of closed loops on the links of some lattice. A more general geometric object central to this paper is the *net*, where "branching" (e.g. trivalent vertices) is allowed, but the geometric objects still do not have ends. Precisely, the ground state $|\Psi\rangle$ can be written as a sum over geometric objects $\mathcal{G}$

$$|\Psi\rangle = \sum_{\mathcal{G}} w(\mathcal{G})|\mathcal{G}\rangle \ . \tag{1}$$

In the quantum loop models discussed below, the weight depends on the number of loops, while for the nets, it will depend on the chromatic polynomial of a (dual) graph.

A tell-tale sign of a ground state described as a sum over geometric objects is that the number of ground states depends on the topology of two-dimensional space. Such behavior is often referred to as "topological order", and such a system is said to be in a "topological phase". For example, in the toric code and in the triangular-lattice quantum dimer model [9], there is only one ground state when space is topologically a sphere, but there are four when space is topologically a torus. The latter is best understood in terms of loops: the four cases correspond to having an even or odd number of loops wrap around each of the two cycles of the torus.

When the ground state has this structure, it is then easy to understand how fractionalized excitations arise. A state with a pair of anyons looks like the ground state (e.g. a sum over configurations with closed loops), except at the locations of the anyons. Each anyon corresponds to the end of a loop or net, so that each pair of anyons is attached by a segment of loop or net. In more picturesque terms, "cutting a loop" results in an anyon pair. Since away from the anyons themselves, the state looks like the ground state, the energy density of such a state is localized around the locations of the anyons. Such a configuration is exactly what one would describe as a two-quasiparticle state. The novelty here is that in each configuration in this sum, the quasiparticles are attached by segments of loop, no matter how far apart they are. This is why it is possible for non-trivial braiding to occur: when quasiparticles are brought around each other, there are strings attached! The segments of loop attached must go through each other when they are attached to particles being braided. When there are multiple states with the same energy and quantum numbers, the braiding is then described by a matrix, opening up the possibility of

non-abelian braiding.

This picture, introduced in [2, 3, 6] and developed further in papers like [10, 11, 12], is very compelling. Moreover, it is not hard to find model Hamiltonians which have the properties described above.

The difficulties in implementing this proposal revolve around ensuring that the anyons are *deconfined*: their energy should not increase as they are moved apart from each other. In terms of the geometric picture, this means that effectively the loop or net segments should have effectively no energy per length. This seems like it should be automatic, given that the Hamiltonian is chosen so that the ground state is a sum over loop configurations of all lengths, and the weights only depend on topological properties of the loops. However, it is well understood in two-dimensional classical loop models how a confinement scale can be introduced. A classical loop model must be critical in order for loops of all length scales to contribute to the partition function. Nevertheless, when the weight per loop exceeds a certain value, it is impossible for there to be a critical point with deconfined loops [13]. Below I refer to the analogous problem in quantum loop models as the $d = \sqrt{2}$ barrier.

In this paper I explain how to go beyond the $d = \sqrt{2}$ barrier by specifying the inner product on the Hilbert space appropriately. This inner product is local and has desirable topological properties. A striking consequence is that the loop models are equivalent to the net models of [11]. Namely, the appropriate inner product for quantum loop models leads naturally to describing the degrees of freedom in terms of Ising spins. Moreover, I will show how in the ground state these Ising spins form nets whose weights depend on a topological quantity, the chromatic polynomial of the dual of the net [11]. The string-net models of [7, 12] are generalizations of these net models; the string-net models have the advantage of being exactly solvable, while the models here have a much simpler Hamiltonian. The exact solution of the string-net models means that one can prove anyons are deconfined there; I will argue here that the same topological phase occurs in the models here.

As discussed above, a prime desire in quantum models with geometric degrees of freedom is to find topological order and fractionalized excitations. Since these arise only from strong correlations, this behavior cannot be understood by using traditional perturbative techniques. Thus the main method of making the problem tractable has been to utilize model Hamiltonians, in the hope that these capture the essential physics of the entire phase.

A key tool in finding a Hamiltonian comes from exploiting the *quantum self-duality* of the net models introduced here. The ground state can be described in terms of geometric degrees of freedom in two separate but equivalent ways, on a lattice and its dual. This is a consequence of the fact that the anyon fusion matrix describes the change of basis to the dual. The quantum self-duality makes it easy to find a Hamiltonian with $|\Psi\rangle$ as its ground state. This Hamiltonian involves only interactions around a vertex and a face, so that for the square lattice this requires only four-spin interactions (thinking of the net variable on each link as a "spin".) This Hamiltonian is therefore considerably simpler than that of other known time-reversal-invariant models with non-abelian statistics in the spectrum. (The string-net models [7], for example, require 12-spin interactions.)

In section 2, I define the quantum loop models considered, and explain precisely what the difficulties with using the naive inner product on the Hilbert space. The modified inner product is described in section 3, where I explain how the classical loop model results of [14] allow the $d = \sqrt{2}$ barrier to be cracked in quantum loop models. In section 4, the map to the net models is given in detail. Several examples are given in section 5, making contact both with the toric code and with string-net models. The concept of quantum self-duality is described in section 6,

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and used to find Hamiltonians with the desired ground state. How quantum-group algebras enter into this picture is explained in section 7. Some generalizations are briefly discussed in section 8. The explicit derivation of the topological weight of the net in terms of the chromatic polynomial is presented in an appendix.

Some of the results of sections 2 and 3 were discussed in arXiv:0711.0014. This paper renders that preprint obsolete, and changes the notation considerably.

2 Quantum loop models and their inner product(s)

As discussed in the introduction, the desired ground state in a topological phase is a sum over geometric objects. The simplest example of such is to make the geometric objects self- and mutually-avoiding *loops* [6]. The Hilbert space of a quantum loop model is spanned by loop configurations on some lattice; each loop configuration $\mathcal{L}$ corresponds to a (not necessarily orthonormal) basis element.

A compelling reason to study quantum loop models is that there is already a well-understood topological quantum field theory with loop degrees of freedom: Chern-Simons theory [15]. Here the loops are the Wilson loops, and it is well understood how to compute their correlators. The braiding of the strands of loop has all the properties for non-abelian anyons [16]. In particular, if one considers doubled Chern-Simons theory, one can obtain a time-reversal-invariant theory. Thus one can hope that a quantum loop model may provide a lattice version of this topological quantum field theory.

The connection between $SU(2)$ Chern-Simons theory and the Jones polynomial [17, 15] suggests not only that loops be the degrees of freedom, but the precise ground-state wavefunction itself [6]. The Jones polynomial is an invariant of knots and links. If two links have different polynomials, then they must be topologically distinct (the converse is not true). To compute the Jones polynomial $J(q)$, one first projects the link down on to a plane, so that braidings become over- or under-crossings. Each crossing then is resolved into a sum over configurations where the lines no longer cross. (Such a sum is particularly natural in the context of a quantum field theory, where the space of projected links is a Hilbert space.) Each configuration then has a value proportional to $d^{n_{\mathcal{L}}}$, where $d = q + q^{-1}$, and $n_{\mathcal{L}}$ is the number of loops in the collection of closed loops $\mathcal{L}$.

These results suggest that one study a quantum loop model where the Hilbert space is spanned by loop configurations $\mathcal{L}$, and whose (unnormalized) ground state is

$$|\Psi\rangle = \sum_{\mathcal{L}} d^{n_{\mathcal{L}}} |\mathcal{L}\rangle \ . \tag{2}$$

This paper focuses entirely on models with this ground state, even when the degrees of freedom of particular lattice realizations of this are rewritten in terms of nets. In section 6 I will provide a Hamiltonian which has this $|\Psi\rangle$ as its ground state.

To have a finite number of ground states on the torus, the parameter d must take on special values [6]. The theory is unitary as well when k is a positive integer, where k is defined by

$$d = 2\cos\left(\frac{\pi}{k+2}\right) \ . \tag{3}$$

In knot-theory language, k rational corresponds to the values where there exists a Jones-Wenzl projector [17, 18], which in the quantum language is a local operator which annihilates the ground state at the corresponding value of d.

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Figure 1: The two-state quantum system at each vertex of the completely packed quantum loop model. The dashed line is the corresponding link on the net lattice $\mathcal{N}$, while the dotted line is that of the dual $\widehat{\mathcal{N}}$.

This section focuses on the "completely packed" loop model, where every link of the lattice is covered by self- and mutually-avoiding loops. Here also I require that there be four links touching each site, so two key examples are the square and Kagomé lattices. It is fairly obvious how to extend these results by relaxing the complete-packing restraint; this generalization will be discussed in section 8.

In this completely packed model, there are two ways that the loops can avoid each other at each vertex. The quantum model therefore is described by a two-state system at every vertex [6, 10], with basis elements $|1\rangle$ and $|\widehat{1}\rangle$ illustrated in fig. 1. Any completely packed loop configuration can be built up by choosing one of these two basis elements at each vertex.

The *loop lattice* is the lattice the loops live on, where all vertices have four links touching them. To define which state is which on a given loop lattice, it is convenient to define other lattices $\mathcal{N}$ and $\widehat{\mathcal{N}}$. For reasons which will later become obvious, $\mathcal{N}$ is called the *net lattice*. The net lattice $\mathcal{N}$ is defined so that the loop lattice is the medial lattice for both $\mathcal{N}$ and its dual $\widehat{\mathcal{N}}$ (a lattice and its dual have the same medial lattice). In an attempt to avoid confusion with the graphs describing nets later on, I will typically refer to the vertices of these lattices as "sites", and the edges of these lattices as "links". With the choice of $\mathcal{N}$, $|1\rangle$ is defined so that the loops do not cross the link of $\mathcal{N}$; $|\widehat{1}\rangle$ is defined so that the loops do not cross $\widehat{\mathcal{N}}$. When the loop lattice is the square lattice, both $\mathcal{N}$ and $\widehat{\mathcal{N}}$ are also square lattices with links $\sqrt{2}$ times longer than the links of the loop lattice. If the loop lattice is the Kagomé lattice, $\mathcal{N}$ and $\widehat{\mathcal{N}}$ are respectively the honeycomb and triangular lattices.

In order to define the Hilbert space of the quantum model, the inner product must be specified. The simplest inner product is to make each loop configuration an orthonormal basis element [6]. This amounts simply to requiring $\langle\widehat{1}|1\rangle = 0$. This inner product, however, is undesirable for several reasons, and so I refer to it as "naive".

The first reason why the naive inner product is undesirable is the "$d = \sqrt{2}$" barrier. Using the naive inner product in the completely packed loop model gives the and the expectation value of a diagonal operator $\mathcal{O}$ in the ground state is

$$\frac{\langle\Psi|\mathcal{O}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\sum_{\mathcal{L}}\mathcal{O}(\mathcal{L})d^{2n_{\mathcal{L}}}}{\sum_{\mathcal{L}} d^{2n_{\mathcal{L}}}} \tag{4}$$

where $\mathcal{O}(\mathcal{L})$ is the value of $\mathcal{O}$ for the loop configuration $\mathcal{L}$. This correlator is exactly that of a *classical* loop model where each loop gets a weight d^2. The reason each loop is weighted by d^2 and not d is of course that probabilities in quantum mechanics are given by the wavefunction's magnitude squared. The normalization

$$\langle\Psi|\Psi\rangle = \sum_{\mathcal{L}} d^{2n_{\mathcal{L}}} \tag{5}$$

is simply the partition function of this classical loop model.

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The completely packed classical loop model has long been studied. It is the $Q = d^4$-state Potts model at its self-dual point: the loops surround clusters in the Fortuin-Kasteleyn expansion [19]. The loops live on the links of the medial lattice for the Potts model lattice, so the square-lattice loop model describes the square-lattice Potts model, while the Kagomé loop model corresponds to the honeycomb or triangular lattice (since the Potts model is at its self-dual point, the two are equivalent). The Potts model at its self-dual point is well understood, and the properties of the loops can be analyzed by Coulomb-gas methods [13]. In fact, the Temperley-Lieb algebra, of central importance in understanding non-abelian braiding and the Jones polynomial, originally was invented to map the square-lattice Potts model to the six-vertex model [20]. An immediate consequence of the map is that the self-dual point is critical only when $Q \leq 4$. In terms of loops, a length scale appears for $Q > 4$ because a large-enough weight per loop favors the creation of many short loops, instead of an ensemble of those of all lengths. For $Q \leq 4$, these completely packed loops are indeed critical. They behave as (i.e. are in the same universality class as) the loops in the $O(N)$ loop model in its dense phase, where $Q = N^2$ [13]. The behavior is the same for honeycomb and triangular lattices as well.

These results for the classical model mean that in the quantum model, the classical loop partition function $\langle \Psi | \Psi \rangle$ is dominated by "short loops" when the weight per loop d^2 is larger than 2 [13]. Thus loops of arbitrarily-long length do not appear in the ground state of the quantum model. Such behavior is not topological, since a length scale appears. This means that anyons are confined when $d > \sqrt{2}$: there is a "$d = \sqrt{2}$ barrier" [6, 10]. Even if one relaxes the requirement of complete packing by allowing links not to have loops on them, the $d = \sqrt{2}$ barrier remains: the "dilute" $O(N)$ model does not have a critical point with all lengths of loops when $N > 2$ [13].

Using the naive inner product is therefore undesirable because of this barrier. Since $k = 1$ has abelian statistics [2], these leaves only one candidate quantum loop model with the naive inner product, that with $k = 2$ and $d = \sqrt{2}$. The loops are indeed critical, but with the naive inner product the quantum model is gapless. A theorem [10] indicates that there cannot be a gap under a broad set of situations. Moreover, it has recently been shown that some correlators of *local* observables decay algebraically as well [21]. A theorem [22] then requires that such a model be gapless, and so quantum critical. Such gapless models may be quite interesting in the context of deconfined quantum critical points [23], but having a gap is necessary for obtaining topological order.

A second compelling reason why the naive inner product is undesirable comes from considering a more general model where the loops may have ends [24]. If one finds a Hamiltonian so that configurations with loop ends have a gap, then as discussed in the introduction, each loop end is a candidate for an non-abelian anyon. Consider two such states with loop ends at the same locations. To obtain a topological theory like Chern-Simons in three-dimensional spacetime, one expects that the inner product of these two states should depend only on topological quantities. The most natural way of defining the inner product is then to "glue" the dangling loop ends of the two together, so that the combined configuration consists entirely of closed loops. For example, let $|\eta\rangle$ and $|\chi\rangle$ each have four loop ends in the same places, but let the loop ends be connected in different ways. Computing the inner product is easily done via the schematic pictures in figure 2. The naive inner product requires $\langle \chi | \eta \rangle$ to vanish. However, since $\langle \chi | \eta \rangle$ corresponds topologically to a single loop, it should not vanish – topologically it is the same as the inner product of a two-anyon configuration with itself. This generalized model thus gives another compelling reason to not require loop configurations to be orthonormal. In the next sections I describe a different inner product which solves the problems discussed in this section.
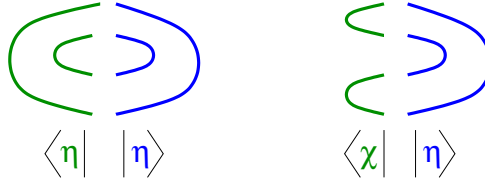
Figure 2: Gluing to find the topological inner product

3 Cracking the $d = \sqrt{2}$ barrier

The discussion in the preceding section illustrates that different loop configurations should not be orthogonal. However, one cannot just impose a topological inner product on the loops like that illustrated in figure 2 by fiat. This would violate locality: knowing whether two points are connected by a strand is inherently a non-local property. Nevertheless, there is a local inner product which allows one to go beyond $d = \sqrt{2}$, while having desirable topological properties.

The simplest generalization of the naive inner product is to keep each loop configuration a basis element of the Hilbert space, but not require them to be orthogonal. In the completely packed loop model, this means that at each vertex

$$\begin{pmatrix} \langle 1|1 \rangle & \langle 1|\widehat{1} \rangle \\ \langle \widehat{1}|1 \rangle & \langle \widehat{1}|\widehat{1} \rangle \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} \tag{6}$$

so that the naive inner product corresponds to $\lambda = 0$. This inner product is positive definite when $|\lambda| < 1$. This inner product can be generalized to the dilute loop model in an obvious fashion.

Keeping the ground state a sum over loops with weight d each (2), the corresponding classical loop partition function is no longer given by (5). With the more general inner product, the classical partition function $\langle \Psi | \Psi \rangle$ is now given by a sum over two completely packed loop configurations $\mathcal{L}$ and $\mathcal{L}'$ on the same lattice, one coming from $|\Psi\rangle$ and the other from $\langle \Psi|$. Letting n_X be the number of vertices at which $\mathcal{L}$ and $\mathcal{L}'$ differ,

$$\langle \Psi | \Psi \rangle = \sum_{\mathcal{L}, \mathcal{L}'} d^{n_{\mathcal{L}} + n_{\mathcal{L}'}} \lambda^{n_X} \ . \tag{7}$$

This classical partition function (7) arising from the new inner product (6) describes two completely packed loop models coupled when $\lambda \neq \pm 1$ [14]. Since the partition function of the Q-state Potts model can be expanded in terms of completely packed loops with weight $\sqrt{Q}$ each, (7) equivalently describes two coupled self-dual d^2-state Potts models. For $\lambda = 1$ and $\lambda = -1$, the decoupled Potts models are at their self-dual ferromagnetic and antiferromagnetic points respectively, which are critical when $d = \sqrt{Q} \leq 2$. However, the inner product is not positive definite when $|\lambda| = 1$, so we cannot choose these values directly. At $\lambda = 0$, the terms in the sum in (7) are non-zero only when $\mathcal{L} = \mathcal{L}'$, so we recover the naive inner product, where loops are weighted by d^2. This is critical only when $d \leq \sqrt{2}$ as described above. At $d = \sqrt{2}$, the coupled loop models are those of the Ashkin-Teller models, where in the spin formulation the model remains critical for all $0 \leq \lambda \leq 1$.

For anyons to be deconfined in the completely packed quantum loop model, the corresponding classical loop model with partition function (7) must be critical for value or values of $|\lambda| < 1$, with loops of all length scales appearing in the partition function. This phase diagram on the

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square lattice has been analyzed in detail in [14], with the fortunate result that there exists a critical phase even when $d > \sqrt{2}$! It was shown there that is a critical point at $\lambda = \lambda_c$, where

$$\lambda_c = -\sqrt{2}\sin\left(\frac{\pi(k-2)}{4(k+2)}\right) \ . \tag{8}$$

Moreover, for $-1 \leq \lambda < \lambda_c$, there is a phase where the loops are critical. In this critical phase, *non-local* correlators of loops decay algebraically. An example of such a correlator is the probability that two segments of loop are on the same loop; this is non-local because one needs to scan the entire system to find if there is a loop connecting the two. In [14], this exponent and its generalization to "watermelon" operators were computed numerically, providing convincing evidence of the critical phase. Note that this critical phase occurs for *negative* values of λ, but as long as $|\lambda| < 1$, the inner product (6) in the quantum model remains positive definite.

For k integer, the coupled Potts models have a local formulation in terms of heights taking values $1, 2, \ldots k + 2$. This arises by using the "RSOS" representation of the Temperley-Lieb algebra [25]. At $\lambda = \lambda_c$, the model is critical in both the height and loop formulations. Changing λ away from λ_c corresponds to a relevant perturbation in both formulations. However, for the loops (but not the heights), this perturbation causes a flow to the decoupled antiferromagnetic critical points at $\lambda = -1$.

Similar behavior also occurs in the toric code. Indeed, below it is shown that the toric-code ground state can be obtained at $d = \sqrt{2}$ in the completely packed loop model. In the toric code, the corresponding classical model is bond percolation. Correlators between individual bonds are trivially non-critical, because the bonds are placed in an uncorrelated fashion on the lattice. However, non-local properties of the loops surrounding clusters of bonds are critical [13].

Thus for $-1 \leq \lambda \leq \lambda_c$, the classical loop model on the square lattice with partition function (7) is critical. The local height model is not critical when $-1 < \lambda < \lambda_c$, so correlators of local operators decay exponentially. The latter is a necessary [22] condition for having a gap. The ground state of the quantum loop model on the square lattice for this region of λ therefore is appropriate for anyonic excitations.

The corresponding classical models on other lattices do not seem to have been studied, with one exception. This is for the "Fibonacci" case $k = 3$, where $d = (1 + \sqrt{5})/2$, the golden mean. In the next section, I will show how this case is related to models discussed in [7, 11, 12]. A beautiful theorem from combinatorics, Tutte's golden identity [26], then implies that this classical loop model is critical [11, 12]. Moreover, in this case, the quantum model is closely connected to an exactly solvable string-net model, which indeed has deconfined anyons [7].

These facts, along with general arguments based on universality, provide good evidence that the classical loop model on other lattices has a critical phase for $d \geq \sqrt{2}$. Thus it seems likely that cracking the barrier should be possible for a variety of lattices, as long as the appropriate inner product is chosen.

4   Nets from loops

In this section I show that with the appropriate choice of inner product, the completely packed quantum loop model is naturally described in terms of *nets*. Rewriting these loop models in terms of nets show that the ground state (2) is almost identical to that of models of [11], and the simplest of the "string-net" models of [7]. It is thus quite likely that their excitations will share similar properties as well.

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A natural choice of λ results from considering the fusion properties of models with the non-abelian anyons, or equivalently, the Wilson loops in Chern-Simons theory. The algebraic structure underlying these fusion properties is discussed in section 7. This choice of λ, however, can be motivated heuristically by extending the discussion at the end of section 2. Consider a generalized model, where loops may have ends, on a loop lattice consisting entirely of one vertex and four links. The two states on this lattice are simply $|1\rangle$ and $|\widehat{1}\rangle$ at the one vertex. These two configurations are topologically equivalent to the four-anyon configurations $|\eta\rangle$ and $|\chi\rangle$ in figure (2). When gluing these together to obtain the inner products, it is natural to expect that the result will be the same as that of the expectation value of the corresponding Wilson loops in Chern-Simons gauge theories in 2+1 dimensional spacetime. Two loops are formed in $\langle\eta|\eta\rangle$ or $\langle\chi|\chi\rangle$, while a single loop is formed in $\langle\eta|\chi\rangle$, so the ratio of the two should be $d^2/d = d$. Utilizing the topological correspondence with with $|1\rangle$ and $|\widehat{1}\rangle$ means that one expects

$$\lambda = \frac{\langle\widehat{1}|1\rangle}{\sqrt{\langle 1|1\rangle\,\langle\widehat{1}|\widehat{1}\rangle}} = \frac{\langle\chi|\eta\rangle}{\sqrt{\langle\chi|\chi\rangle\,\langle\eta|\eta\rangle}} = \pm\frac{1}{d}\ .$$

In section 3 I explained how taking λ negative is necessary to crack the $d = \sqrt{2}$ barrier. This suggests that the inner product (6) be defined by taking

$$\lambda = -\frac{1}{d}\ . \tag{9}$$

This choice of inner product has remarkable consequences, resulting in a beautiful topological and algebraic structure when the completely packed loop model is rewritten in terms of nets. Even more importantly, deconfined anyons are possible for d larger than $\sqrt{2}$. As shown in [14] and discussed here in section 3, the corresponding classical loop model is critical on the square lattice whenever $-1 \le \lambda \le \lambda_c$, with λ_c given in (8). This with $\lambda = -1/d$, the loops are critical for $k \le 6$, i.e. $d \le 2\cos(\pi/8)$. For $k < 6$, $\lambda = -1/d$ falls in the region where correlators of local quantities decay exponentially. At $k = 6$, $\lambda_c = 1/d$, so the local degrees of freedom are critical as well; this should therefore describe a deconfined quantum critical point.

4.1   The two orthonormal bases

The first step in finding nets is rewrite the loop model in an orthonormal basis. The states $|0\rangle$ and $|\widehat{0}\rangle$ orthogonal to $|1\rangle$ and $|\widehat{1}\rangle$ respectively are

$$|0\rangle = \frac{1}{\sqrt{d^2 - 1}}\left(d|\widehat{1}\rangle + |1\rangle\right)\ , \tag{10}$$

$$|\widehat{0}\rangle = \frac{1}{\sqrt{d^2 - 1}}\left(d|1\rangle + |\widehat{1}\rangle\right). \tag{11}$$

This indeed yields $\langle 0|1\rangle = \langle\widehat{0}|\widehat{1}\rangle = 0$ and $\langle 1|1\rangle = \langle\widehat{1}|\widehat{1}\rangle = 1$. These degrees of freedom live on the sites of the loop lattice.

There are therefore two natural orthonormal bases for the Hilbert space at each site, one with basis elements $(|0\rangle, |1\rangle)$, and the other with basis elements $(|\widehat{0}\rangle, |\widehat{1}\rangle)$. The unitary transformation relating the vector spaces spanned by these basis elements is implemented by the matrix

$$F = \begin{pmatrix} \langle\widehat{0}|0\rangle & \langle\widehat{0}|1\rangle \\ \langle\widehat{1}|0\rangle & \langle\widehat{1}|1\rangle \end{pmatrix} = \frac{1}{d}\begin{pmatrix} 1 & \sqrt{d^2 - 1} \\ \sqrt{d^2 - 1} & -1 \end{pmatrix}\ . \tag{12}$$

This particular matrix F is familiar in studies of non-abelian anyons [1, 4]. It arises, for example, when describing the fusion of the anyons in the Read-Rezayi series of fractional quantum Hall states [27]. In terms of the corresponding conformal field theory, this is the fusion matrix for two spin-1/2 primary fields in $SU(2)_k$.

The correspondence of the change-of-basis matrix with the fusion matrix is not a coincidence: at the beginning of this section it is noted how a single vertex can be thought of as a four-anyon state in a generalized model. The fusion matrix for non-abelian anyons is also a change-of-basis matrix, relating different bases for four-anyon states. Consider four anyons at fixed positions in the identity channel. Being in the identity channel means that the combined state of all four is bosonic: braiding all four around anything will not change the wave function. An essential requirement for non-abelian statistics is that the multi-anyon Hilbert space be degenerate, so the Hilbert space is spanned by orthonormal basis states labeled by the index j. One way of specifying the basis is to pick two of the anyons and let j label the possible channels of these two anyons, i.e. j labels how the combination or "fusion" of these two anyons will behave under braiding. Since the overall channel is the identity, the other two anyons will necessarily be in the channel conjugate to j. Labeling the four anyons by k, l, m and n going clockwise from the bottom left, the basis found by fusing k and l is schematically represented on the left-hand side of figure 3. Of course, one can choose two different anyons to define the basis; the basis found by fusing k and n is shown on the right-hand side of figure 3. Any choice will result in an acceptable basis, and

Figure 3: The four-anyon fusion matrix

since they all describe the same four-anyon Hilbert space, there must be a unitary transformation relating any two of them. The "fusion matrix" $F_{ij}^{(klmn)}$ is the matrix implementing this change. This fusion can be understood quite elegantly in terms of quantum-group algebras, and will be explained in more depth in section 7.

Treating the change-of-basis matrix F as a four-anyon fusion matrix provides the key to rewriting the ground state of completely packed loops in terms of nets. One orthonormal basis for the Hilbert space of this four-anyon state is then given by $(|0\rangle, |1\rangle)$, whereas another is $(|\widehat{0}\rangle, |\widehat{1}\rangle)$, the same basis for each vertex in the completely packed loop model. The change-of-basis matrix F of the latter is then indeed the four-anyon fusion matrix. Since the fusion matrix has a nice geometric interpretation given in figure 3, this implies a similar geometric description for each orthonormal basis of the loop model.

Precisely, the geometric degrees of freedom in the $(|0\rangle, |1\rangle)$ basis live on the links of the net lattice $\mathcal{N}$ defined at the beginning of section 2. Those in the $(|\widehat{0}\rangle, |\widehat{1}\rangle)$ basis live on the links of the dual $\widehat{\mathcal{N}}$ of the net lattice. On $\mathcal{N}$, one simply denotes the state $|1\rangle$ by putting a segment of net on the corresponding link, while $|0\rangle$ is denoted by leaving it empty. On $\widehat{\mathcal{N}}$, one does the same for $|\widehat{1}\rangle$ and $|\widehat{0}\rangle$ respectively. The original loop lattice can be forgotten, because complete packing

requires all its links to be covered: all the degrees of freedom live on the links of the net lattice (or equivalently, its dual).

A set of orthonormal basis elements for the model can be specified by specifying whether each link has a $|1\rangle$ or $|0\rangle$ on it; graphically this corresponds respectively to covering the link or not. Thus the each element $|E\rangle$ of this orthonormal basis corresponds to a graph E, whose edges are the covered links of $\mathcal{N}$, and whose vertices are the sites of $\mathcal{N}$ touching these links. The example corresponding to all states $|1\rangle$ on the sites of the Kagomé lattice is drawn in figure 4; the graph E here completely covers the net lattice $\mathcal{N}$, which is honeycomb here. Likewise, each state written in the dual basis ($|\widehat{0}\rangle, |\widehat{1}\rangle$) can be represented as a graph with edges a subset of links of $\widehat{\mathcal{N}}$. The example corresponding to all states $|\widehat{1}\rangle$ on the sites of the Kagomé lattice is drawn in figure 5, when $\widehat{\mathcal{N}}$ is the triangular lattice.
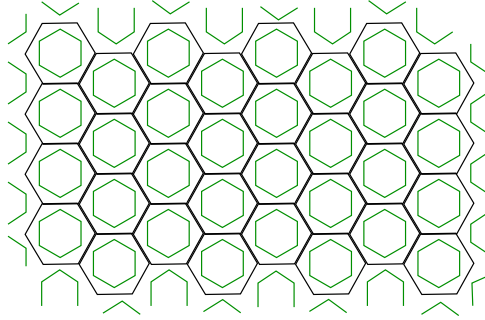

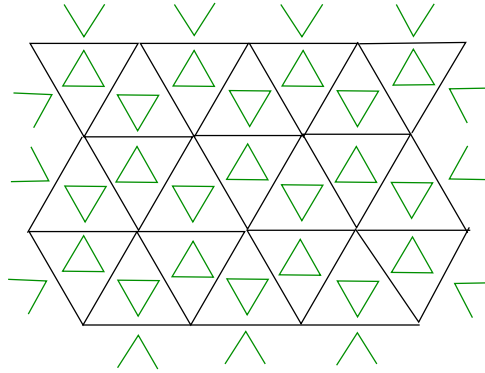Figure 4: How the configuration with all states $|1\rangle$ is described as a net on $\mathcal{N}$

Figure 5: How the configuration with all states $|\widehat{1}\rangle$ is described as a net on $\widehat{\mathcal{N}}$

4.2   The ground state as a sum over nets

In these orthonormal bases, the ground state (2) in these bases has a beautiful topological description in terms of *nets*. A net is a geometric object without ends, i.e. no vertex has just one edge touching it. The beautiful part is that only nets appear in the ground state: $\langle E|\psi\rangle = 0$ whenever E has an end.

To prove this, and to also find the topological weight of each net in the ground state, let $|\mathcal{L}\rangle$ be the state of the system written in the loop basis, i.e. in terms of $|1\rangle$ and $|\widehat{1}\rangle$. Since the loop

basis is not orthonormal, it is useful to introduce the state $\langle\overline{\mathcal{L}}|$ such that for $\mathcal{L} \neq \mathcal{L}'$,

$$\langle\overline{\mathcal{L}}|\mathcal{L}'\rangle = 0, \qquad \langle\overline{\mathcal{L}}|\mathcal{L}\rangle = 1$$

For the case at hand,

$$\langle\overline{1}| = \frac{d}{\sqrt{d^2-1}}\langle\widehat{0}|, \qquad \langle\overline{\widehat{1}}| = \frac{d}{\sqrt{d^2-1}}\langle 0| . \tag{13}$$

By construction, the weight of a particular loop configuration $\mathcal{L}$ in any state $|\psi\rangle$ is then $\langle\overline{\mathcal{L}}|\psi\rangle$. In particular, the weight in the ground state $|\Psi\rangle$ from (2) is

$$\langle\overline{\mathcal{L}}|\Psi\rangle = d^{n_\mathcal{L}} . \tag{14}$$

Inserting a complete set of states can now be done in the loop basis:

$$\sum_{\mathcal{L}} |\mathcal{L}\rangle\langle\overline{\mathcal{L}}| = 1 .$$

The weight of an orthonormal basis element in the ground state $|\Psi\rangle$ is then given by

$$\begin{aligned}
\langle E|\Psi\rangle &= \sum_{\mathcal{L}}\langle E|\mathcal{L}\rangle\langle\overline{\mathcal{L}}|\Psi\rangle \\
&= \sum_{\mathcal{L}}\langle E|\mathcal{L}\rangle d^{n_\mathcal{L}}
\end{aligned} \tag{15}$$

It is now straightforward to prove that if E has any ends, this amplitude $\langle E|\Psi\rangle$ vanishes. Let E_{end} be any edge configuration with an end at some vertex. By definition, the states on the edges around this end are of the form $\langle 100\ldots|$. Thus $\langle E_{end}|$ has non-vanishing inner product with only two states in the loop basis: $|1\widehat{1}\widehat{1}\ldots\rangle$ and $|\widehat{1}\widehat{1}\widehat{1}\ldots\rangle$. Let $|\mathcal{L}_1\rangle$ and $|\mathcal{L}_2\rangle$ be two loop configurations given by the former and the latter around the vertex with the end, and otherwise identical. Using $\langle 1|\widehat{1}\rangle = -1/d$ means the inner products are then related by

$$\langle E_{end}|\mathcal{L}_1\rangle = -\frac{1}{d}\langle E_{end}|\mathcal{L}_2\rangle . \tag{16}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ for the square lattice are illustrated in figure 6. $\mathcal{L}_1$ and $\mathcal{L}_2$ are topologically

Figure 6: The two loop configurations $\mathcal{L}_1$ and $\mathcal{L}_2$ with inner products related in (17).

identical, except that the latter has an extra loop. Thus $n_{\mathcal{L}_2} = n_{\mathcal{L}_1} + 1$, and

$$\langle E_{end}|\mathcal{L}_1\rangle d^{n_{\mathcal{L}_1}} + \langle E_{end}|\mathcal{L}_2\rangle d^{n_{\mathcal{L}_2}} = 0 .$$

All configurations with a non-vanishing inner product with $\langle E_{end}|$ can be grouped in pairs of the form of $\mathcal{L}_1$ and $\mathcal{L}_2$. Putting this information together with (15) means that

$$\langle E_{end}|\Psi\rangle = 0 \tag{17}$$

The ground state is indeed a sum over edge configurations without ends, i.e. nets!

4.3 The topological weight

The (non-vanishing) weight $\langle N|\Psi\rangle$ for each net can be found by writing the net N as a sum over loops as in (15). I will show here that this sum can be used to find a much simpler expression for it in terms of the chromatic polynomial. Moreover, in this formulation it is obvious how the weight depends on the topological properties of the nets.

The graph N associated with each net is defined as the graph whose edges are those in $\mathcal{N}$ which have the state $|1\rangle$ on them, such that no vertex in $\mathcal{N}$ has only a single $|1\rangle$ touching it. The vertices of N are those vertices of $\mathcal{N}$ with at least two edges in N touching them (i.e. vertices of $\mathcal{N}$ with only states $|0\rangle$ touching are not included in N).

To find the simpler expression for the weight of each net $\langle N|\Psi\rangle$, it is useful to rewrite $\langle N|$ in terms of the states in (13):

$$\langle 1| \;\; = \;\; \langle \overline{1}| - \frac{1}{d}\langle \overline{\widehat{1}}| \tag{18}$$

$$\langle 0| \;\; = \;\; \frac{\sqrt{d^2-1}}{d}\langle \overline{\widehat{1}}| \; . \tag{19}$$

Because $\langle \overline{1}|$ does not appear on the right-hand side of (19), $\langle N|\mathcal{L}\rangle$ is non-vanishing only for loops with $|\widehat{1}\rangle$ on every edge in $\mathcal{N}$ not part of the net graph N.

For edges part of N, (18) shows that both kids of loop states contribute to the weight. Moreover, they contribute in exactly the same combination which arose in [28, 29] in studies of the Potts model and the chromatic polynomial (see e.g. figure 6 in [28] or figure 8 in [29]). In particular, in [29, 30] it is shown that the chromatic polynomial of the dual graph $\widehat{N}$ can be computed by summing over the loop configurations created by the substitution of the combination (18) !

The precise expression for the weight of each net in the ground state can be derived using a slight variation of Lemma 2.5 in [29]. The derivation for $\mathcal{N}$ (and thus N) a planar graph is presented in the appendix; the result is

$$\langle N|\Psi\rangle = \alpha \left(\frac{1}{\sqrt{d^2-1}} \right)^{L_N} \chi_{\widehat{N}}(d^2) \tag{20}$$

where L_N is the number of edges of the net N, i.e. what in statistical mechanics would usually be called its "length", α is an unimportant overall constant (40) given in the appendix, and $\chi_{\widehat{N}}(d^2)$ is the *chromatic polynomial* for the dual graph $\widehat{N}$. The chromatic polynomial vanishes for any graph N which is not a net, so (20) remains valid even if N were not a net, i.e. one recovers (17) as well.

The chromatic polynomial only depends on topological properties of N. Put picturesquely, if one treats the edges of N as borders separating countries, $\chi_{\widehat{N}}(Q)$ is the number of ways of coloring each country with Q colors such that adjacent countries are not colored the same. Typically, chromatic polynomials are described in terms of the dual graph (the reason for the $\widehat{N}$ in the subscript of χ). The dual graph $\widehat{N}$ is defined with a vertex corresponding to each face of N, and with an edge connecting two vertices for each edge in N separating the two faces. Thus when Q is an integer, $\chi_{\widehat{N}}(Q)$ counts the number of ways of coloring $\widehat{N}$ such that any two vertices connected by an edge have different colors.

In statistical mechanics, the chromatic polynomial arises from the low-temperature expansion of the Q-state Potts model. (The completely packed loops discussed above arise from the Fortuin-Kasteleyn expansion [19].) The nets corresponds to domain walls in the Potts model, separating

regions of different Potts spins. If the nets live on the links of $\mathcal{N}$, then these spins, taking values $1, 2 \ldots Q$, live on the sites of the dual lattice $\widehat{\mathcal{N}}$. By definition, the chromatic polynomial then counts the number of different spin configurations possible for a given domain wall configuration. Thus the classical partition function of the Potts model can be written as a sum over nets as

$$Z_{Potts} = \sum_N K^{L_N} \chi_{\widehat{N}}(Q) \tag{21}$$

where K is the weight per unit length of the domain wall. In the usual Potts language, $K \to 0$ corresponds to zero temperature and ferromagnetic, $K \to \infty$ to zero temperature and antiferromagnetic, and $K = 1$ to infinite temperature.

It is obvious that the number of colorings for any finite graph is a polynomial (with integer coefficients) in Q, so it can be evaluated for all values of Q, not just integers. A common way of computing $\chi_{\widehat{N}}(Q)$ for all $Q = d^2$ is to use contraction-deletion relation on $\widehat{N}$: see e.g. [31] or [11]. An equivalent way to define it directly in terms of N for all d^2 is described in [29, 30]. This method is convenient for the study of quantum loop models, because it involves only topological properties of N. One can find $\chi_{\widehat{N}}$ by repeatedly applying the following relations to simplify the graph, and then using the fact that $\chi_{\widehat{\emptyset}}(d^2) = d^2$ for the empty graph $\emptyset$. First, as illustrated in figure 7, if any subgraph of N is a "tadpole", $\chi_{\widehat{N}}$ must vanish. Second, the chromatic polynomial for a graph
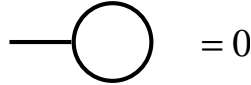

Figure 7: Nets with tadpoles do not appear in the ground state.

with a 4-valent vertex can be reduced to two trivalent vertices by using the relation illustrated in figure 8. Since there are two ways of doing this, this relation gives an identity involving pairs of
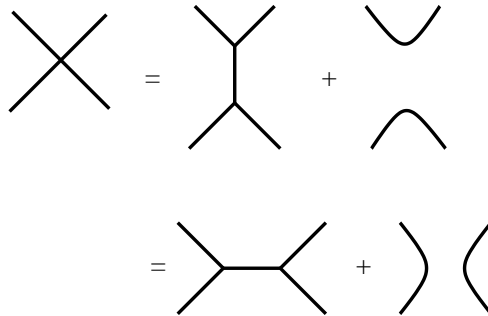

Figure 8: Relations among chromatic polynomials

trivalent vertices as well. Finally, any simple closed curve in N (i.e. any subgraph of N which is topologically a loop without intersections) can be removed if one multiplies the resulting graph by $d^2 - 1$. All of these are relations are valid for *any subgraph* of N, meaning that they can be applied locally without regard to the rest of the graph. Proofs of these relations, and the generalization to vertices with more legs, are easy to find by using the coloring description. Thus for example, for "theta" graph illustrated in figure 9, $\chi_{\widehat{\theta}}(d^2)/d^2 = 0 + (d^2 - 1)^2 - (d^2 - 1) = (d^2 - 2)(d^2 - 1)$.
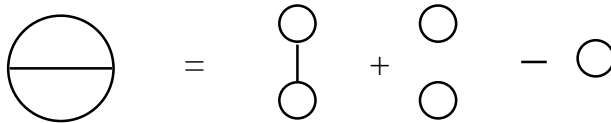
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Figure 9: Evaluating the theta graph by using the relations in figures 7 and 8

The upshot is that the ground state in the orthonormal net basis is

$$|\Psi\rangle = \alpha \sum_N \left(\frac{1}{\sqrt{d^2-1}}\right)^{L_N} \chi_{\widehat{N}}(d^2)|N\rangle \tag{22}$$

This is the ground state for the models discussed at length in [11], once the weight per unit length of net is fixed to be $1/\sqrt{d^2-1}$.

A weight per unit length not equal to 1 seems to violate the requirement that the weight be topological. However, as discussed in section 3, for $d \leq 2\cos(\pi/8)$, the partition function of the associated classical loop model $\langle\Psi|\Psi\rangle$ on the square lattice contains contributions from loops of all sizes [14]. Thus (22) shows that (at least for the square lattice) this weight is desirable. Moreover, allowing "dilution" for the loops (i.e. relaxing the complete packing requirement) is an *irrelevant* perturbation. This is characteristic of a "dense" phase of the classical loop model [13], where once the weight per unit length is large enough, the system is dominated by configurations covering the lattice. The weight per unit length becomes therefore becomes irrelevant in this phase. Although the weights per unit length of the loops and nets are not the same, they are closely related: diluting loops results in diluting nets. It is thus reasonable to assume that the weight per unit length of the net in (22) is irrelevant, as long as it is large enough (order 1).

5   Examples of topological order

5.1   An abelian case: the toric code

The first two non-trivial values of k provide some illuminating examples. For $k = 2$ $(d = \sqrt{2})$ on the square lattice, the ground state is quite familiar: it is that of the toric code [2]. The weight $\chi_{\widehat{N}}(2) = 2$ if $\widehat{N}$ is bipartite, and is zero if it is not. Thus no nets with trivalent vertices can appear in the ground state, and each vertex in $\mathcal{N}$ must have 0, 2 or 4 adjacent edges in N. Moreover, the weight per unit length for $d = \sqrt{2}$ is 1, so each allowed net has the same weight. The ground state is the equal-amplitude sum over all nets with no trivalent vertices. This is indeed the ground state for the toric code. Configurations where the net has ends, or three edges touching a vertex, are necessarily excitations. Although these are fractionalized (they have mutual fermion statistics [2]), they are abelian, a consequence of the fact of the each allowed net has the same weight.

On general lattices, when $k = 2$, the nets in the ground state must have an even number of edges in N at each vertex, and each such net gets equal weight. When $k = 2$ and the net lattice is honeycomb, the ground state is filled with loops only. Even though the original non-orthonormal loops on the Kagomé lattice have weight $\sqrt{2}$ per loop, these quantum loops on the net lattice have an orthonormal inner product with weight one per loop. This corresponds to the $O(1)$ loop model on the honeycomb lattice. The phase diagram of this model has been extensively studied [13]; for a weight per unit length greater than $1/\sqrt{3}$, the loops are in a dense phase. The dense phase is critical, with loops on average covering almost all the lattice. Since the weight per unit

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length for the loops on the net lattice here is 1, the orthonormal loops here are indeed critical as necessary for topological order.

When $k = 2$, the corresponding classical model is simply the Ising model at infinite temperature on the dual lattice; the nets are the domain walls separating regions of unlike Ising spins. This is the special case $Q = d^2 = 2$ of the more general result for the low-temperature expansion of the Potts model discussed above; just because it is called the low-temperature expansion does not mean it is not valid at infinite temperature. Because of the infinite temperature, correlators of local objects have correlation length effectively zero. However, for all lattices studied (like square and honeycomb already mentioned), correlators of non-local objects like nets decay algebraically. These correlators belong to the same universality class as percolation models. It is thus natural to conjecture that on any lattice, this model at $k = 2$ has abelian topological order. In section 8, I will show that it is possible to get non-abelian topological order by allowing vacancies into the (original) loop model.

As a quick check on this picture, consider a generalization with a variable weight per unit length for nets in the ground state. In the corresponding classical Ising model for the ground state, this corresponds to varying the temperature. I argued above that varying this coupling is irrelevant, so that the nets remain critical. However, a quantum critical point occurs when this weight is infinity and the net lattice is the honeycomb lattice. Here this forbids configurations where a site has no net touching it, and so yields the quantum dimer model, where the "dimer" is on the single link touching each site without a net. The nets indeed remain critical in the corresponding classical model, but the local degrees of freedom (e.g. the dimers) also become critical here. The quantum dimer model on the honeycomb lattice is therefore at a quantum critical point, as is well known [32]. In the classical model, varying the coupling from infinity is a relevant perturbation, but while the local degrees of freedom flow to the trivial infinite-temperature point, the non-local loops remain critical (but with different critical properties). In loop model language, this is known as the flow from the compact or fully packed phase to the dense phase; see e.g. [33]. This thus provides a nice non-trivial check that the quantum model does indeed possess non-abelian topological order.

5.2   A non-abelian case: Fibonacci anyons

Non-abelian topological order occurs at the next-simplest point $k = 3$, where d is the golden mean $\tau = 2\cos(\pi/5) = (1 + \sqrt{5})/2$. Anyons with the fusion matrix (12) with $d = \tau$ are known as "Fibonacci anyons" because the number of n-anyon states is the nth Fibonacci number (see e.g. [4]). Here the model is closely related to one of the "string-net" models of [7], known there as the $N = 1$ string-net model. Since this string-net model is exactly solvable, this provides a valuable check on the results here.

In the string-net model of relevance here, there is a two-state quantum system on every link of the honeycomb lattice, corresponding to a segment of string net covering it or not. The ground state does not include configurations with a site touched by only a single segment of string net, so the ground state is a sum over nets identical to those here. The weight of each string net in the ground state is determined indirectly by demanding that the weights of different configurations be related by the appropriate fusion matrix. Here the fusion matrix is (12) with $d = \tau$, although now all the configurations involved are nets, not the combination of loops and nets as illustrated in figure 3. An explicit expression for the weight of each string net configuration in the ground state was derived in [12]. It involves the chromatic polynomial as in [11] and here, but instead of including weight per unit length, it has a weight $\tau^{3/4}$ for each trivalent vertex in the net, i.e.

the number of sites of the net lattice $\mathcal{N}$ which have all three links touching them. Denoting the ground state of the string-net model to be $|\Psi_s\rangle$, equation (24) of [12] gives

$$\langle N|\Psi_s\rangle = \tau^{3V_{tri}/4}\chi_{\widehat{N}}(\tau^2)/\tau^2 \tag{23}$$

for V_{tri} trivalent vertices.

A Hamiltonian can be found so that the string-net model is exactly solvable, so it has been proved that the ground state has topological order, and that the excited states are non-abelian anyons [7]. The close similarity of the ground state with that of the net models developed here means that it is likely both have the same physics, even with the much simpler Hamiltonian found in the next section. One strong argument in favor of this follows from the discussion in section 3: the classical loop model for $k = 3$ on the square lattice is in a critical phase, and there is no evidence that the honeycomb lattice behaves otherwise.

A second argument for topological order and deconfined anyons comes from a remarkable property of the chromatic polynomial at $k = 3$, the *golden identity* [26, 29]. This identity involves the *square* of the chromatic polynomial at a particular argument:

$$\chi_{\widehat{N}}(\tau + 2) = (\tau + 2)\,\tau^{3\,V_{tri}/2-4}\,(\chi_{\widehat{N}}(\tau^2))^2 \ . \tag{24}$$

when N involves at most trivalent vertices and closed loops. When computing correlators in the quantum theory, each configuration in the orthonormal net basis is weighted by the square of (23). Thus the golden identity gives the remarkable simplification [12]

$$|\langle N|\Psi_s\rangle|^2 = (\tau + 2)\chi_{\widehat{N}}(\tau + 2)$$

so that e.g.

$$\langle \Psi|\Psi\rangle = (\tau + 2)\sum_N \chi_{\widehat{N}}(\tau + 2) \ .$$

Comparing with (21) means that the corresponding classical partition function is simply that of a *single* $Q = (5 + \sqrt{5})/2$ Potts model at infinite temperature. Correlators in the ground state of the quantum model are those of this single classical Potts model. Moreover, since this value of Q is less than 4, for an appropriate value of the weight per unit length K, this model has a critical point. Here, $K_c = 0.51117...$ [34]). Thus the ground state $|\Psi_q\rangle$ is quantum critical when

$$\langle N|\Psi_q\rangle = \tau^{3V_{tri}/4}K_c^{L_N/2}\chi_{\widehat{N}}(\tau^2) \tag{25}$$

Since the weight per unit length 1 in (23) is greater than $\sqrt{K_c}$, the classical Potts model is indeed in its high temperature phase, where the domain walls should be critical.

The models considered here have a ground state defined by (20), which differs from that of the quantum-critical ground state (25) and the string-net ground state (23) in two ways. It has weight 1 instead of weight $\tau^{3/4}$ per trivalent vertex, and has weight $\tau^{-1/2}$ instead of weight 1 per unit length. Changing the weight per unit length corresponds to changing the temperature in the classical Potts model, so this perturbation is relevant at the critical point. Since $\tau^{-1/2} > \sqrt{K_c}$, this perturbation here pushes the model toward the high-temperature phase, where local degrees of freedom have exponentially-decaying correlations, but the loops and nets are critical. There are no exact results for the consequence of changing the weight per trivalent vertex. However, the honeycomb lattice is bipartite, and if one instead includes a weight for a trivalent vertex whenever it appears on half of the sites while leaving the weight on the other half unchanged, it is known that there exists a critical self-dual line [35]. Thus the corresponding operator in a

field-theory description is irrelevant at the critical point (there are no marginal perturbations in the Potts models for $Q < 4$). The perturbing operator for the changing the weight on the other half of the vertices should have the same dimension, so this operator should be irrelevant as well. Thus a weight per trivalent vertex is irrelevant in the classical model.

The golden identity coupled with these results from classical Potts models therefore gives a second argument implying that in the quantum model when $k = 3$ and the loop lattice is Kagomé, the ground state $|\Psi\rangle$ from (20) has topological order. Since the topological order is the same as that of the string-net ground state $|\Psi_q\rangle$, it is reasonable to conjecture that there exists a Hamiltonian such that the excited states over $|\Psi\rangle$ are non-abelian anyons.

To conclude the discussion of the $k = 3$ case on the honeycomb lattice, it is interesting to understand why the ground state (20) for $d = \tau$ and the string-net ground state (23) are slightly different (even though the above arguments indicate their physics is the same). The latter arises by demanding that the fusion matrix for the *nets* be unitary. The fusion matrix for nets F^{net} relates linear combinations of weights of nets in the ground state. In the $k = 3$ case, F^{net} relates the weights of the configurations given in figure 3, if all the thin green lines are replaced with thick net lines. The reason F^{net} is two-by-two when $k = 3$ is clear from the quantum-group language discussed below in section 7: the spin-2 representation when $k = 3$ is reducible, so two spin-1 representations here can fuse only to spin 0 or spin 1. (This statement can be phrased graphically via the Jones-Wenzl projector [11, 29].)

The fusion matrix for nets can be computed here by using the expression of the weights (22) in terms of the chromatic polynomial. Using the relation in figure (8) and the Jones-Wenzl projector for the chromatic polynomial at $k = 3$ [11, 29] shows how the ground-state weights depend on the topology of the nets being related. Since F^{net} relates nets with different lengths, the weight per unit length in (22) causes it to depend on the lengths L_v and L_h of the internal vertical and horizontal lines on the right- and left-hand sides of figure 3. Since $d^2 - 1 = \tau$ for $k = 3$,

$$F^{\mathrm{net}}(L_h, L_v) = \frac{1}{\tau} \begin{pmatrix} 1 & \tau^{2+L_v/2} \\ \tau^{-1-L_h/2} & -\tau^{(L_h-L_v)/2} \end{pmatrix} . \tag{26}$$

This matrix is not unitary, but it does satisfy the necessary consistency requirement

$$F^{\mathrm{net}}(L_h, L_v) F^{\mathrm{net}}(L_v, L_h) = 1 .$$

While demanding a unitary F for nets is certainly a reasonable requirement (and seems necessary to make the string-net Hamiltonian constructed in [7] hermitian), it does not seem to be a necessary one. The fusion matrix (12) was constructed to change between orthonormal bases, and so in this context must be unitary. This fusion matrix for the nets F^{net} need not be unitary, because it is not a change of basis: it describes a topological property of the ground state written in terms of nets. Consistency in this context requires only that $F^{\mathrm{net}}(L_h, L_v) F^{\mathrm{net}}(L_v, L_h) = 1$. However, when one modifies the preceding calculation to compute $F^{\mathrm{string\ net}}$ from the string-net ground-state weights given by (23), the factors of L_h and L_v do not appear, and the weight per trivalent vertex rescales the off-diagonal terms to give $F^{\mathrm{string\ net}} = F$ from (12). This indeed is unitary, and moreover makes the application of the golden identity quite natural [12].

The crucial property of the ground state is that in the continuum, there be no confinement scale, and I have argued at length above that this holds even with the weight per unit length of the nets in the ground state. This implies that it is perfectly consistent to require that the weights of the nets in the ground state be related by using F^{net} in (26) instead of F. The advantage of relaxing the requirement of unitarity is that it is much easier to find a Hamiltonian

with $|\Psi\rangle$ as its ground state than it is to find one with $|\Psi_s\rangle$. In the next section 6, I describe this Hamiltonian.

6 Quantum self-duality and the Hamiltonian

In this section I show that a key advantage of making nets from loops is that such a net model is *quantum self-dual*. This means that there are two different natural ways of writing the ground state $|\Psi\rangle$, one on the net lattice $\mathcal{N}$ as described in section 4.3, and the other on the dual $\widehat{\mathcal{N}}$ of the net lattice. The quantum self-duality makes it easy to find a Hamiltonian having $|\Psi\rangle$ as its ground state. On the square lattice, such a Hamiltonian only requires four-spin interactions.

The quantum self-duality of the quantum net model described in this paper follows almost immediately from the preceding analysis. The choice of which lattice is the net lattice $\mathcal{N}$, and which is the dual $\widehat{\mathcal{N}}$ is arbitrary: both have the loop lattice as their medial lattice. By definition, exchanging $\mathcal{N}$ with $\widehat{\mathcal{N}}$ simply corresponds to exchanging the state $|1\rangle$ with $|\widehat{1}\rangle$, and so $|0\rangle$ with $|\widehat{0}\rangle$. Thus a *dual-lattice net* D can be defined in the same way as the net N. Namely, the edges of D live on the links on the dual lattice $\widehat{\mathcal{N}}$, with each edge corresponding to the state $|\widehat{1}\rangle$. This dual-lattice net is called D to avoid confusion with $\widehat{N}$, the dual of a net N on $\mathcal{N}$.

The weight $\langle D|\Psi\rangle$ of the net D in the ground state is therefore given by a computation identical to that which led to the expression (20). All one needs to do is exchange the basis ($|0\rangle$, $|1\rangle$) with ($|\widehat{0}\rangle, |\widehat{1}\rangle$); the two bases have equivalent properties. Rerunning the derivation in the appendix of the weight in this fashion gives

$$\langle \widehat{D}|\Psi\rangle = \alpha' \left(d^2 - 1\right)^{-L_D/2} \chi_{\widehat{D}}(d^2) \tag{27}$$

Here L_D is the length of the dual net D, i.e. the number of edges covered by the state $|\widehat{1}\rangle$. The unimportant constant α' is given by replacing $\mathcal{N}$ with $\widehat{\mathcal{N}}$ in expression (40) for α in the appendix.

It is important to emphasize that the ground state $|\Psi\rangle$ in (27) is the *same* ground state being discussed all along. This ground state can be written as a sum over nets N on $\mathcal{N}$, or as sum over nets D on the dual lattice $\widehat{\mathcal{N}}$. The weight of each state in each of these sums is defined in the same fashion. For this reason I refer to ground state as being *quantum self-dual*. Note that it is essential to include the weights per unit length in (20,27) to have this quantum self-duality.

This quantum self-duality implies non-trivial relations among chromatic polynomials. For example, by using the F matrix change of basis, $\chi_{\widehat{N}}$ can be rewritten as a sum over chromatic polynomials of edge subgraphs of the dual graph. Namely, consider a case where $N = \mathcal{N}$, i.e. every link of the net lattice $\mathcal{N}$ is covered. When an edge in N is replaced with a sum over unfilled and filled edges of $\widehat{\mathcal{N}}$ by using the F matrix, the entries of the F matrix (12) give an unfilled edge a factor $-\sqrt{d^2 - 1}$ times that of the filled one. Thus

$$\chi_{\widehat{N}}(d^2) \; \propto \; \langle N|\Psi\rangle \; \propto \; \sum_D (-\sqrt{d^2-1})^{-L_D} \langle D|\Psi\rangle \; \propto \; \sum_D (1-d^2)^{-L_D} \chi_{\widehat{D}}(d^2) \; .$$

where the proportional signs can be replaced with the actual constants with a little effort.

It is useful to check this quantum self-duality by rederiving this identity using the duality of partition function of the classical Potts model (the Tutte polynomial in mathematical language). Classical Potts duality written in the low-temperature expansion (21) is

$$Z_{Potts} = \sum_N K^{L_N} \chi_{\widehat{N}}(d^2) \propto \sum_D (K_D)^{L_D} \chi_{\widehat{D}}(d^2)$$

where as before D is a net graph on the dual $\widehat{\mathcal{N}}$ of the net lattice, and the weight per unit length of D is related to that of N by

$$\left(\frac{1}{K_D} - 1\right)\left(\frac{1}{K} - 1\right) = d^2 .$$

This duality holds for any d^2 and any net graph $\mathcal{N}$. The chromatic polynomial for a given net N can be found by taking the (zero-temperature antiferromagnetic) limit $K \to \infty$, so that the sum over N reduces to the graph where $N = \mathcal{N}$. When $K \to \infty$, then $K_D = 1/(1 - d^2)$, so indeed

$$\chi_{\widehat{N}}(d^2) \propto \sum_D (1 - d^2)^{-L_D} \chi_{\widehat{D}}(d^2) .$$

This weight per unit length is exactly what arises from exploiting quantum self-duality. This illustrates the quantum self-duality is quite different from classical self-duality. (Classical self-duality is the fact that there is a value where $K = K_D \equiv K_c$ so that the Potts model on $\mathcal{N}$ at $K = K_c$ is equivalent to the Potts model at $K_D = K_c$ on $\widehat{\mathcal{N}}$.)

The quantum self-duality makes it easy to find a Hamiltonian H with $|\Psi\rangle$ as its ground state. The trick is to find local projection operators that annihilate $|\Psi\rangle$. Any such operator can be included in H; since by construction $H|\Psi\rangle = 0$, and eigenvalues of projection operators are non-negative, $|\Psi\rangle$ is then a ground state. Such Hamiltonians in this context are usually referred to as being of Rokhsar-Kivelson type [9]. It is essential to include enough terms in H so that when space is topologically a sphere or annulus, $|\Psi\rangle$ is the only ground state. This should be the case if the space of states is connected by H, meaning that by acting with H enough, one can reach any state in the Hilbert space from any other.

The marvelous consequence of quantum self-duality is that it gives enough projectors to make H connected. This is because any state included in the ground state can be geometrically described as a net on both the net lattice $\mathcal{N}$ *and* on the dual lattice $\widehat{\mathcal{N}}$. Thus any projection operator which annihilates all nets N on $\mathcal{N}$ *or* annihilates all nets D on $\widehat{\mathcal{N}}$ also annihilates $|\Psi\rangle$.

Define the projection operators P_0 and P_1 acting on each link in the $(|0\rangle, |1\rangle)$ basis as

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{28}$$

Let V be a vertex of $\mathcal{N}$, and $e_{V1}, e_{V2}, \dots$ be the edges touching this vertex. Then let

$$\mathcal{P}_V = P_1(e_{V1})P_0(e_{V2})P_0(e_{V3}) \dots + \text{ rotations} \tag{29}$$

where $P_i(e)$ acts on the quantum two-state system on the edge e, and the $\dots$ represent P_0 acting on any remaining edges around V. In other words, each term in $\mathcal{P}_V$ has P_1 acting on one edge touching V and P_0 on the others. Therefore $\mathcal{P}$ annihilates any net, since by definition a net includes no vertices with only one $|1\rangle$ touching them. The analogous operators $\mathcal{P}_{\widehat{V}}$ can be defined in the $(|\widehat{0}\rangle, |\widehat{1}\rangle)$ basis, and because of the quantum self-duality, also must annihilate $|\Psi\rangle$.

In the $(|0\rangle, |1\rangle)$ basis, the operators $\mathcal{P}_V$ are diagonal, so a Hamiltonian comprised solely of these is not sufficient; they annihilate any net. The key to finding a Hamiltonian with a unique ground state is to observe that the statement that $\mathcal{P}_{\widehat{V}}|\Psi\rangle = 0$ is independent of basis: $\mathcal{P}_{\widehat{V}}$ transformed to the $(|0\rangle, |1\rangle)$ basis still annihilates $|\Psi\rangle$. Thus the operators $P_{\widehat{0}}$ and $P_{\widehat{1}}$, defined by

$$P_{\widehat{0}} = FP_0F = \frac{1}{d^2}\begin{pmatrix} 1 & \sqrt{d^2 - 1} \\ \sqrt{d^2 - 1} & d^2 - 1 \end{pmatrix}$$

$$P_{\widehat{1}} = FP_1F = \frac{1}{d^2}\begin{pmatrix} d^2 - 1 & -\sqrt{d^2 - 1} \\ -\sqrt{d^2 - 1} & 1 \end{pmatrix},$$

project onto the states $|\widehat{0}\rangle$ and $|\widehat{1}\rangle$ respectively in the $(|0\rangle, |1\rangle)$ basis. Each vertex of $\widehat{\mathcal{N}}$ corresponds to a face of $\mathcal{N}$. Letting $e_{F1}, e_{F2}, \ldots$ be the edges around the face F, define

$$\mathcal{P}_F = P_{\widehat{1}}(e_{F1})P_{\widehat{0}}(e_{F2})P_{\widehat{0}}(e_{F3})\ldots + \text{ rotations } . \tag{30}$$

Each of the operators $\mathcal{P}_V$ and $\mathcal{P}_F$ annihilates $|\Psi\rangle$. Each is a projection operator, so it has only non-negative eigenvalues. In the $(|0\rangle, |1\rangle)$ basis, each $\mathcal{P}_V$ is diagonal, but $\mathcal{P}_F$ is not; in fact, every entry of the latter is non-zero. When rewritten in the original (non-orthonormal) loop basis, all these operators contain off-diagonal terms. Therefore the Hamiltonian

$$H = \sum_V \mathcal{P}_V + \sum_F \mathcal{P}_F \tag{31}$$

likely has all the desired properties for topological order and non-abelian anyons in the spectrum. Exploiting the quantum self-duality is essential to finding this H.

It is illuminating to understand how the Hamiltonian acts in the loop basis. The matrix elements are found via

$$\langle \overline{\mathcal{M}}|H|\mathcal{L}\rangle = \sum_{m,n} \langle \overline{\mathcal{M}}|m\rangle \langle m|H|n\rangle \langle n|\mathcal{L}\rangle$$

so that the change of basis matrix R for each site of the loop lattice is

$$R = \begin{pmatrix} \langle 0|1\rangle & \langle 0|\widehat{1}\rangle \\ \langle 1|1\rangle & \langle 1|\widehat{1}\rangle \end{pmatrix} = \frac{1}{d}\begin{pmatrix} 0 & \sqrt{d^2-1} \\ d & -1 \end{pmatrix} .$$

R is not unitary because the loop basis is not orthonormal.

Rewriting $\mathcal{P}_V$ in this basis gives an upper-right-triangular matrix which acts non-trivially on only two kinds of loop configurations, one of the form $|\widetilde{111}...\rangle$ around the vertex, and the other $|1\widehat{1}1...\rangle$. For the square lattice, these are the two types of configurations pictured in figure 6. Rewriting $\mathcal{P}_F$ in the loop basis exchanges $|1\rangle$ with $|\widehat{1}\rangle$, but since this acts around a face of the net lattice, the two types of loops related are still precisely those in figure 6. Moreover, for both $\mathcal{P}_V$ and $\mathcal{P}_F$, the relative coefficient of the two types of terms is always related. Namely, since

$$R^{-1}P_1R = \begin{pmatrix} 1 & -\frac{1}{d} \\ 0 & 0 \end{pmatrix}$$

then for any $\mathcal{M}$,

$$-\frac{1}{d}\sum_{\mathcal{L}_1} \langle \mathcal{M}|H|\mathcal{L}_1\rangle = \langle \mathcal{M}|H|\mathcal{L}_2\rangle ,$$

where the sum is over all possibilities for $\mathcal{L}_1$, i.e. all locations of the $|1\rangle$ edge around the vertex. Since $\mathcal{L}_2$ has one more loop than $\mathcal{L}_1$, for a state $|\psi\rangle$ to be annihilated by H,

$$\langle \overline{\mathcal{L}_2}|\psi\rangle = d\langle \overline{\mathcal{L}_1}|\psi\rangle \tag{32}$$

for each possible $\mathcal{L}_1$. Thus indeed $|\Psi\rangle$ is annihilated by H.

This rewriting of H to act on loops gives more than a check that $|\Psi\rangle$ is a ground state: it gives a proof that $|\Psi\rangle$ is the unique ground state when space is topologically a sphere. It is straightforward to see that *any* two loop configurations on the sphere can be related by repeatedly doing "surgery" between configurations of the type in figure 6 [6, 10]. (This is not true on the torus, because this surgery will not change the number of loops around the torus.) In the ground

states, (32) must be obeyed for any two configurations related by a single surgery. So repeating this surgery means all relative coefficients are determined, leaving $|\Psi\rangle$ as the unique ground state on the sphere.

I do not know how to prove that this Hamiltonian has a gap, but it seems very plausible. The first term in (31) gives an energy to net ends, while the second does likewise for any configuration with a dual-lattice net end. This does not prove the model is gapped (bear in mind the theorem of [10]), because a net end is not an exact eigenstate. Nevertheless, the ground state is that of the toric code in the $k = 2$ case and almost identical to that of the string-net models for $k = 3$. Moreover, local variables in the corresponding classical models have exponentially decaying correlators for $k < 6$. Thus it is reasonable to conjecture that this Hamiltonian is gapped.

To understand this Hamiltonian in more depth, it is useful to consider the special case $d = \sqrt{2}$, where $k = 2$. As discussed in section 5, for this case, the ground state $|\Psi\rangle$ is identical to that of the toric code. However, the Hamiltonian is not, and the reason is illuminating.

The Hamiltonian for the toric code is conveniently written in terms of Pauli matrices acting on the $(|0\rangle, |1\rangle)$ basis. On the square lattice it is

$$H_{toric} = \frac{1}{2}\sum_V (1 - \sigma^z_{V1}\sigma^z_{V2}\sigma^z_{V3}\sigma^z_{V4}) + \frac{1}{2}\sum_F (1 - \sigma^x_{F1}\sigma^x_{F2}\sigma^x_{F3}\sigma^x_{F4}) \ .$$

The vertex terms ensure that the ground state is a sum over nets. The face terms are non-diagonal, but map nets to nets, and so ensure that the ground state is a sum over nets with equal amplitudes. The face terms only can change the length L_N of the nets by an even number. Thus on surfaces of genus ≥ 1, the face terms only can change the number of nets wrapped around a cycle by an even number, leaving two possible ground states for each cycle.

To relate this to the Hamiltonian (31), it is useful to recall a fact about $|\Psi\rangle$ noted in section (5): the net contains no trivalent vertices when $k = 2$. Thus each projector

$$\mathcal{P}^{JW}_V = P_0(e_{V1})P_1(e_{V2})P_1(e_{V3})P_1(e_{V4}) + \text{ rotations} \tag{33}$$

annihilates $|\Psi\rangle$. This is known as the Jones-Wenzl projector for $k = 2$ [17, 18], and was discussed at length in this context in [29]. Of course, the same arguments apply to the dual nets as well, so

$$\mathcal{P}^{JW}_F = P_{\hat{0}}(e_{F1})P_{\hat{1}}(e_{F2})P_{\hat{1}}(e_{F3})P_{\hat{1}}(e_{F4}) + \text{ rotations} \tag{34}$$

also annihilates $|\Psi\rangle$. Since these have non-negative eigenvalues, adding them to the Hamiltonian preserves $|\Psi\rangle$ as a ground state. Using

$$2P_0 = 1 + \sigma^z, \qquad 2P_1 = 1 - \sigma^z, \qquad 2P_{\hat{0}} = 1 + \sigma^x, \qquad 2P_{\hat{1}} = 1 - \sigma^x,$$

gives

$$H_{toric} = \sum_V (\mathcal{P}_V + \mathcal{P}^{JW}_V) + \sum_F (\mathcal{P}_F + \mathcal{P}^{JW}_F) = H_{k=2} + \sum_V \mathcal{P}^{JW}_V + \sum_F \mathcal{P}^{JW}_F \ . \tag{35}$$

Thus H_{toric} differs from H simply by adding in the Jones-Wenzl projectors.

There exists a unique Jones-Wenzl projector at any value of k (see the appendix by Goodman and Wenzl in [6], and for the net case, [29]). Thus an interesting question is if adding Jones-Wenzl projectors allows a model with the same ground state to be solved for other values of k. For higher values of k, the Jones-Wenzl projector requires involving more nets, so the Hamiltonians will require interactions with more spins. For $k = 3$, one can find such terms involving six spins

on the square lattice, but the closely-related exactly solvable Hamiltonian of [7] hints that 12-spin interactions may be necessary to find a solvable model.

As opposed to H, the Hamiltonian H_{toric} is exactly solvable, and it is easy to see that $|\Psi\rangle$ is the unique ground state on the sphere. The proof above indicates that this remains true for H, so that this proves that the Hamiltonian of the toric code can be deformed without changing the ground state on the sphere. An interesting open question is whether the excited states remain anyonic and gapped after the deformation, but the arguments above indicate that this is a good possibility. Moreover, the existence of Jones-Wenzl projectors is a good sign that a gapped topological phase is possible for any k. Since one can add a Jones-Wenzl projector to the Hamiltonian and preserve $|\Psi\rangle$ as the ground state, it seems highly unlikely that one can add such a term with arbitrary coefficient without the model having a gap. In fact, this is essentially what the string-net models do; their Hamiltonian includes the Jones-Wenzl projector.

7   Quantum-group algebras in quantum loop and net models

The structure of the loop and net models discussed above can be understood very naturally in terms of *quantum group algebras*. This approach also illuminates why it is desirable to choose the inner product as above.

To follow this, one doesn't need to know much more than a paragraph's worth of results on quantum groups, so these results are presented here. For an excellent in-depth discussion in a closely related context, see [36]. A quantum-group algebra $U_q(G)$ is a one-parameter (q) deformation of a Lie algebra G. The algebra $U_q(G)$ has representations whose tensor-product rules corresponding to representations of G, with one major distinction: if q is a root of unity, all but a finite number of these are reducible. The quantum loop and net models discussed above are related to $U_q(SU(2))$, with $q = e^{i\pi/(k+2)}$; only the spin-$j/2$ representations with $j = 0, 1, \ldots k+1$ are irreducible. The Jones-Wenzl projector discussed above annihilates the ground state precisely because of this truncation; for the appropriate value of k, the spin-$(k+1)/2$ representation can be decomposed into representations of the irreducible ones, and so there is a linear relation among ground-state configurations.

To give a few simple but important examples, the tensor product of two spin-$1/2$ representations of $U_q(SU(2))$ with $k \geq 2$ is decomposed into irreducible representations as

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \ ; \tag{36}$$

for $k = 1$, the truncation means that the 1 does not appear. A spin-1 representation has a tensor product with another spin 1 like that of ordinary $SU(2)$ when $k \geq 4$:

$$1 \otimes 1 = 0 \oplus 1 \oplus 2 \qquad \text{for } k \geq 4; \tag{37}$$

for $k = 2, 3$ the 2 does not appear, and for $k = 2$ the 1 does not appear as well. The Jones-Wenzl projector for $k = 2$ has already been given in (33). In quantum-group language it arises because $1 \otimes 1 \otimes 1 = 1 \otimes 0 = 1$ when $k = 2$, so that three spin-1 representations do not have the identity in their tensor product.

The relation of quantum nets to quantum-group algebras with is that each *type* of geometric degree of freedom in the former corresponds to a *representation* of the latter. Fusion in the former corresponds to a tensor product in the latter, with the fusion matrix comprised of the $6j$ symbols. The (chiral) primary fields of the conformal field theory and the types of Wilson loops of (undoubled) Chern-Simons theory are in one-to-one correspondence with these representations.

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In the $SU(2)_k$ theories considered above, each segment of loop corresponds to the spin-1/2 representation of $U_q(SU(2))$. The requirement that these strands form closed loops then can be understood as requiring that at each vertex, all four segments must fuse to the identity representation. Label the four strands (or equivalently, the four spin-1/2 representations) going clockwise around a vertex in the completely packed loop model $\alpha, \beta, \gamma, \delta$. Consider two of them, say α and β, on opposite sides of the link of the net lattice going through this vertex. Because of (36), these two can fuse either to the spin-0 identity representation, or the spin-1 representation. The other two states, γ and δ, also can fuse to spin 0 or spin 1. The only way for all four to fuse to spin 0 is then for γ and δ to fuse in the same way that α and β did. There are thus two states in the quantum loop model, corresponding to these two choices. These are precisely the states $|0\rangle$ and $|1\rangle$ defined above, thus explaining the choice of notation. This procedure is no different from finding singlets of four $SU(2)$ spins.

Instead of fusing first the pair α and β, one of course could have fused α and δ first. The same arguments apply, but since this is a rotation by 90 degrees of the earlier situation, the states must be rotated as well. Thus this way of fusing results in the states $|\widehat{0}\rangle$ and $|\widehat{1}\rangle$. These two states are not independent of $|0\rangle$ and $|1\rangle$, but are merely another basis. In the $SU(2)_k$ conformal field theory, or in level k $SU(2)$ Chern-Simons field theory, the fusion matrix (12) is the unitary matrix transforming between these two bases. Thus setting $\lambda = -1/d$ in (6) gives precisely the inner product that preserves the quantum-group structure.

Understanding these fusion rules explains why the $k = 2$ is equivalent to the toric code. The net degrees of freedom are spin 1, and the fusion rule for $k = 2$ is $1 \otimes 1 = 0$. This is abelian, because there is only one irreducible state on the right-hand-side. Likewise, for $k = 3$, the truncation means that fusion rule is $1 \otimes 1 = 0 + 1$, characteristic of the fusion of Fibonacci anyons. The truncation also explains why F^{net} in (26) is 2×2 instead of 3×3.

It is important to note that the quantum-group algebra is *not* a symmetry algebra of the model: the loops are not in a spin-1/2 (i.e. two-dimensional) representation of $U_q(SU(2))$. Rather, they *correspond* to the spin-1/2 representation, meaning for example that the loops obey the same fusion rules as do fields in this representation of the $SU(2)_k$ conformal field theory.

This picture gives an elegant way of characterizing a ground state. The unique feature of a ground state is that *at any site of the loop, net or dual net lattices*, the corresponding degrees of freedom on the links touching the vertex must *must fuse to the identity representation*. With the way the model was built, this is automatic for sites of the loop lattice. For the net or dual net lattice, what this means precisely is that the ground state is annihilated by any projector onto a fused state orthogonal to the identity. The projectors in the Hamiltonian (31) are of course examples of this, projecting onto the spin-1 fused state on vertices of the net and dual net lattices.

The types of anyonic excitations themselves also correspond to representations of the quantum-group algebra in the same way that the geometric degrees of freedom do. Any time a state is not annihilated by one of the aforementioned projectors, it by construction is part of an excited state. This is the precise meaning of "cutting" the net. There will then be anyons corresponding to all irreducible representations of the quantum group that can be built up by taking tensor products of the representations corresponding to the nets. Thus even though there are half-integer-spin representations of $U_q(SU(2))$, in the model discussed above only integer-spin anyons occur because segments of nets themselves correspond to the spin-1 representation.

8 Generalizations and Conclusions

This paper introduces and develops models on any lattice whose ground state is a sum over all net configurations. When $k < 6$ and the lattice is square, the work of [14] indicates that correlators of non-local objects like nets in the ground state decay algebraically, while correlators of local objects decay exponentially. Although this does not guarantee that the model is in a gapped non-abelian topological phase, this is a necessary condition. The similarities between these models and the string-net models on the honeycomb lattice also provide substantial evidence for topological order.

One new aspect of the work here is the concept of quantum self-duality, which made it possible to construct a simpler Hamiltonian annihilating the ground state. The key to finding a quantum self-dual ground state was to ensure that the weight of each configuration on the loop lattice, even if not orthonormal, came from the corresponding link invariant. In the case above, the link invariant was the Jones polynomial, but one can construct link invariants (see e.g. [37]) for any $U_q(G)$. Then combining this ground state with the orthonormal basis constructed using the F matrix allows the weights of each orthonormal basis element in the ground state to be found. Since by construction, the nets will still fuse to the identity representation at each vertex, the quantum-group structure should ensure that a nice Hamiltonian with this ground state acting on the nets can be found.

The quantum net models developed here arise from completely packed loops on the medial lattice. There are a number of ways of relaxing this restriction, culminating in the string-net models of [7], where each allowed state on a link corresponds to a primary field in a rational conformal field theory. These generalizations are straightforward to make by utilizing the quantum-group algebras discussed in section 7, and following the work of e.g. [37] and [7]. One specifies the loop lattice, and then which representation(s) of some quantum group $U_q(G)$ appear on which links. Thus for example, in [7], each link of the honeycomb lattice can be in any representation of the quantum-group algebra. The quantum-group algebra will then provide a fusion matrix which allows an orthonormal basis to be determined, the guiding principle being that different representations at the same place (site or link) must be orthogonal. The ground state involves only configurations which fuse to the identity at every vertex. The associated link invariants then provide the topological part of the weight of each configuration in the ground state.

Following this strategy, it appears possible to construct a model with non-abelian anyons at $d = \sqrt{2}$ as well as at larger values. This is done by allowing links of the loop lattice to be empty, i.e. allowing the spin-0 representation of $U_q(SU(2))$ as well as the spin-1/2. Then there are degrees of freedom on both the links and sites of the loop lattice. The ground state will still remain a sum over loops with a weight d per loop, as in (2), only now loops are not completely packed. The inner product at sites with all four edges covered by loops will also remain the same. The corresponding classical loop model will essentially the same, only now there will be a fugacity governing the amount of dilution. Since dilution is irrelevant in the classical coupled loop models [14], it seems likely that this quantum model will be in a deconfined phase for $k < 6$ like the completely packed version. The interesting feature of this model over the completely packed version is that there are excitations of spin 1/2 (and 3/2), at sites on the loop lattice with an odd number of loops touching them. The spin-1/2 ones are non-abelian even at $d = \sqrt{2}$, because the fusion algebra (36) remains non-trivial here.

One can of course generalize the models of this paper without relaxing the complete packing constraint by still requiring that each link of the loop lattice be in the same representation of $U_q(SU(2))$, but using a representation with spin greater than 1/2. For spin 1, the fusion

relation (37) then requires that there be a three-state quantum system at each site (when $k > 3$). Following the analysis of [28, 11, 29, 30] shows that the "loops" here will be nets, with weights given by the chromatic polynomial.

This construction does not guarantee quantum self-duality, nor does it guarantee that correlators in the ground state will have the desired properties for topological order. However, the fact that it works (at least for $k < 6$) in the above models, and always works in the string-net models, means it is very reasonable to hope that this construction of simple(r) Hamiltonians will work in more general settings.

A   Deriving the topological weight

In this appendix I adapt Lemma 2.5 of [29] to show how the weight of each net in the ground state can be expressed in terms of the chromatic polynomial. As in (15), inserting a complete set of states gives $\langle N|\Psi\rangle$ as a sum over loops:

$$\langle N|\Psi\rangle = \sum_{\mathcal{L}} \langle N|\mathcal{L}\rangle \, d^{n_{\mathcal{L}}} \ . \tag{38}$$

As discussed above, because of (19), $\langle N|\mathcal{L}\rangle$ is nonzero only when the states on all edges of $\mathcal{N}$ not part of the net N are fixed. The sum over loops here therefore reduces to a sum over subgraphs N_s of N, depending which of the terms in (18) contributes on each edge. Each N_s has the same vertices as N, but a subset of edges, defined as those where the *second* term in (18) contributes. Each N_s therefore corresponds to a single loop configuration. Therefore

$$\langle N|\Psi\rangle = \sum_{N_s \subseteq N} \left(-\frac{1}{d}\right)^{E(N_s)} \left(\frac{\sqrt{d^2 - 1}}{d}\right)^{E(\mathcal{N}) - E(N)} d^{n_{\mathcal{L}_s}} \tag{39}$$

where $E(G)$ and $V(G)$ are the number of edges and vertices in the graph G, and $n_{\mathcal{L}_s}$ is the number of loops in the unique loop graph determined by N_s.

The next task is therefore to relate the number of loops $n_{\mathcal{L}}$ to topological properties of N_s. Because the loop state is always $|\widehat{1}\rangle$ on all edges not part of N, each vertex of $\mathcal{N}$ not part of N is surrounded by a single loop of minimum length. Thus the number of loops surrounding vertices of $\mathcal{N}$ not in N is $V(\mathcal{N}) - V(N)$. It is straightforward to see by drawing pictures (see figure 12 in [29]) that the remaining loops are the boundary of a regular neighborhood of $\widehat{N_s}$, the dual graph of N_s. The total number of such loops is $k(s) + n(s)$, where $k(s)$ is the number of connected components of $\widehat{N_s}$, and $r(s)$ is the rank of the first homology of $\widehat{N_s}$ (i.e. the number of independent cycles). Therefore

$$n_{\mathcal{L}_s} = V(\mathcal{N}) - V(N) + k(s) + n(s) \ .$$

This can be simplified by noting that $k(s) - n(s) + E(\widehat{N_s}) = V(\widehat{N})$. This is proved by first noting that it is true for the case where $\widehat{N_s}$ has no edges, so that $k(s) = V(\widehat{N})$ and $n(s) = E(\widehat{N_s}) = 0$.

Adding an edge to $\widehat{N}_s$ then either decreases $k(s)$ by 1 (if it connects two previously disconnected clusters), or it increases $n(s)$ by 1 (if it adds an edge connecting two already-connected vertices, it increases the number of independent cycles by 1). Thus the identity follows by induction, giving

$$n_{\mathcal{L}_s} = V(\mathcal{N}) - V(N) + 2k(s) + E(N_s) - V(\widehat{N}) \ ,$$

where I exploited the fact that the edges on a graph are in one-to-one correspondence with those in the dual graph so that $E(\widehat{N}_s) = E(N_s)$.

Plugging this in to (39) gives

$$\langle N|\Psi\rangle = d^r \left(d^2 - 1\right)^{(E(\mathcal{N}) - E(N))/2} \sum_{N_s \subseteq N} (-1)^{E(N_s)} \, d^{2k(s)} \ ,$$

where

$$r = V(\mathcal{N}) - E(\mathcal{N}) + E(N) - V(N) - V(\widehat{N}) \ .$$

Each vertex of a graph corresponds to face of the dual graph, so $V(\widehat{N}) = F(N)$, where $F(G)$ is the number of faces of G. Euler's relation $V(G) - E(G) + F(G) = 2$ then gives

$$r = -F(\mathcal{N}) \ .$$

Finally, this can be related to the chromatic polynomial by using the identity for the chromatic polynomial of any planar graph (see e.g. [31])

$$\chi_{\widehat{N}}(d^2) = \sum_{\widehat{N}_s \subseteq \widehat{N}} (-1)^{E(\widehat{N}_s)} \, d^{2k(s)} \ .$$

This sum over edge subgraphs $\widehat{N}_s$ of $\widehat{N}$ is equivalent to the sum over subgraphs N_s of N, since the edges of a graph and its dual are in one-to-one correspondence. Thus setting

$$\alpha = d^{-F(\mathcal{N})}(d^2 - 1)^{E(\mathcal{N})/2} \tag{40}$$

gives the inner product (20) for any planar net N. As a quick check on this result, this computation can be done easily for the square lattice by utilizing the Temperley-Lieb algebra. This is done at the end of section 2 of [14], where the weight $1/\sqrt{d^2 - 1}$ per unit length of net is indeed recovered.

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